2.2: Sets

Question 1. A certain class consists of 21 students. Of these, 10 plan to major in mathematics and 13 plan to major in computer science. Five students are not planning to major in either subject. How many students plan to major in both subjects? Draw a diagram to explain your reasoning.

Question 2. Explain the distinction between the following two scenarios:

> *Scenario A:* A professor chooses three students to work together on a problem.
> *Scenario B:* A professor asks three students in sequence, and each time calls on a student to answer.

How are the professor's choices constrained in each scenario? Compare and contrast.

Sets: George Cantor described a set as

"any collection into a whole of definite and separate objects of our intuition or of our thought."

Mathematically, a <u>set</u> is a collection of objects called <u>elements</u>. We write $x \in S$ to say x is an element of the set S. We can describe a set by listing out all the elements

$$S = \{x_1, x_2, \dots, x_n\}$$

or by describing a subset which contains elements of the set with certain properties in set-builder notation

$$\{x \in S | x \text{ has property } p\}.$$

Example 1. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then $2 \in A$ and $9 \notin A$. If

$$B = \{x \in A | x \text{ is odd}\},$$

then $B = \{1, 3, 5, 7\}$.

Example 2. Certain "blackboard" letters are used to represent frequent sets. We let $\mathbb{Z}$ represent the integers and $\mathbb{N}$ represent the positive integers and refer to them as the *natural* numbers. Notice $0 \in \mathbb{Z}$ and $0 \notin \mathbb{N}$. We use $\mathbb{R}$ for the set of real numbers and $\mathbb{Q}$ for the set of rational numbers.

Example 3. Let P be the set of all polygons. So P contains all triangles, squares, pentagons, etc. If c is a circle, then $c \notin P$. We could describe the set H of all hexagons in set builder notation as

$$H = \{x \in P | x \text{ has six sides}\}.$$

Sets are convenient for describing groups of objects that are related by some common property. Sometimes one property implies another, for example, all integers are real numbers; i.e.

$$(\forall x)(x \in \mathbb{Z} \to x \in \mathbb{R}).$$

In this case, we say that "$\mathbb{Z}$ is a subset of $\mathbb{R}$" or "$\mathbb{R}$ contains $\mathbb{Z}$" and write $\mathbb{Z} \subseteq \mathbb{R}$. It is easy to see that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Example 4. The *empty set* $\emptyset$ is the set that contains no elements. Therefore, the empty set is a subset of every set, that is, $\emptyset \subseteq X$ for all X. This is because the statement $x \in \emptyset$ is false for any x, so the implication

$$(\forall x)(x \in \emptyset \to x \in X)$$

must be true.

Connection to Logic: You may have noticed that $\subseteq$ has basically replaced $\to$ from logic. So what will replace the connectives $\vee$, $\wedge$, $\neg$, and $\leftrightarrow$? The <u>union</u> $A \cup B$ of two sets A and B is the set of all elements of both A and B put together; i.e.

$$A \cup B := \{x | (x \in A) \vee (x \in B)\}.$$

The <u>intersection</u> $A \cap B$ is the set of all elements that have A and B in common; i.e.

$$A \cap B := \{x | (x \in A) \wedge (x \in B)\}.$$

The <u>complement</u> A' is the set of all elements (in some domain U) which are not in A; i.e.

$$A' := \{x \in U | x \notin A\}.$$

Equality $A = B$ replaces $\leftrightarrow$.

Example 5. Let the following sets be given.

$X = \{n \in \mathbb{Z} | n = 2k \text{ for some odd integer } k\}$
$F = \{n \in \mathbb{Z} | n = 4k \text{ for some integer } k\}$
$E = \{n \in \mathbb{Z} | n \text{ is even }\}$

1. Prove that $F \subseteq E$.

2. Prove that $X = E \cap F'$.

Beyond our introduction to logic: Another useful set to consider is the cartesian product $A \times B$. This consists of all ordered pairs with the first elements coming from A and the second from B; i.e.

$$A \times B := \{(a, b) | (a \in A) \wedge (b \in B)\}.$$

If we wish to have larger sequences, we can consider triples, quadruples, etc; i.e.

$$A_1 \times A_2 \times \cdots \times A_n := \{(a_1, a_2, \ldots, a_n) | \wedge_{i=1}^{n} a_i \in A_i\}$$

is the set of all ordered n-tuples where the ith item comes from A_i. Two ordered pairs (tuples) are equal if and only if their corresponding parts are equal; i.e.

$$(a, b) = (c, d) \iff a = c \text{ and } b = d.$$

One last useful set is the power set $P(S)$ of a set S. This contains all subsets of the set S; i.e.

$$P(S) := \{X | X \subseteq S\}.$$

Example 6. Suppose you want to form a study group with some of the other students in your class. If S is the set of all students in your class, then $P(S)$ is the set of all possible study groups. The empty set would represent the decision to not form a study group at all!

Example 7. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, and suppose the universal set is $U = \{1, 2, \ldots, 10\}$. Find

- $A \cup B$

- $A \cap B$
- B'
- $(A \cap B) \times (A \cap B)$
- $P(A \cap B)$

All of the equivalence and inference rules from the previous chapter have analogues in sets. I have let you space to write them all down in the language of set theory. I encourage you to do this! Especially De Morgan's Laws!!

One more handy rule is the inclusion-exclusion principle. This requires us to define the cardinality (or size) of a set, which we denote by $|X|$.

Example 8. Let S be the set containing all members of the U.S. House of Representatives. Then $|S| = 435$.

The inclusion-exclusion principle states that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example 9. Do Question 1 again with this new knowledge.

Homework. (Due Oct 17, 2018) Section 2.2: 2, 8, 10, 13, 27

Practice Problems. Section 2.2: 1, 3, 4, 5, 11, 12, 15, 16, 21, 22, 26, 28

Equivalence Rules

Name	**Logical Equivalence**	**Set Theory Law**
double negation	$p \iff \neg\neg p$	$A = (A')'$
implication	$p \to q \iff \neg p \vee q$	$A \subseteq B \iff B' \subseteq A'$
De Morgan's laws	$\neg(p \vee q) \iff \neg p \wedge \neg q$ $\neg(p \wedge q) \iff \neg p \vee \neg q$	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$
commutativity	$p \vee q \iff q \vee p$ $p \wedge q \iff q \wedge p$	$A \cup B = B \cup A$ $A \cap B = B \cap A$
associativity	$p \wedge (q \wedge r) \iff (p \wedge q) \wedge r$ $p \vee (q \vee r) \iff (p \vee q) \vee r$	$(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$
distributive	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Inference Rules

Name	**Logical Inference**	**Set Theory Law**
conjunction	$\left.\begin{array}{c} p \\ q \end{array}\right\} \Rightarrow p \wedge q$	$A \cap B = A \cap B$
modus ponens	$\left.\begin{array}{c} p \\ p \to q \end{array}\right\} \Rightarrow q$	$(x \in A) \wedge (A \subseteq B) \Rightarrow x \in B$
modus tollens	$\left.\begin{array}{c} \neg q \\ p \to q \end{array}\right\} \Rightarrow \neg p$	$(x \in B') \wedge (A \subseteq B) \Rightarrow x \in A'$
simplification	$p \wedge q \Rightarrow p$	$A \cap B \subseteq A$
addition	$p \Rightarrow p \vee q$	$A \subseteq A \cup B$